Fuchsian analysis of $\mathbf{S}^2 \times \mathbf{S}^1$ and $\mathbf{S}^3$ Gowdy spacetimes

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Abstract

The Gowdy spacetimes are vacuum solutions of Einstein's equations with two commuting Killing vectors having compact spacelike orbits with $\mathbf{T}^3$, $\mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{S}^3$ topology. In the case of $\mathbf{T}^3$ topology, Kichenassamy and Rendall have found a family of singular solutions which are asymptotically velocity dominated by construction. In the case when the velocity is between zero and one, the solutions depend on the maximal number of free functions. We consider the similar case with $\mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{S}^3$ topology, where the main complication is the presence of symmetry axes. We use Fuchsian techniques to show the existence of singular solutions similar to the $\mathbf{T}^3$ case. We first solve the analytic case and then generalise to the smooth case by approximating smooth data with a sequence of analytic data. However, for the metric to be smooth at the axes, the velocity must be 1 or 3 there, which is outside the range where the constructed solutions depend on the full number of free functions. A plausible explanation is that in general a spiky feature may develop at the axis, a situation which is unsuitable for a direct treatment by Fuchsian methods.

1 Introduction

The singularity theorems by Hawking and Penrose show that in general spacetimes, singularities will be present [21]. Apart from existence, these results give very little information on the nature of these singularities.

There is however a general proposal for the structure of the generic cosmological singularity, put forth by Belinskii, Khalatnikov and Lifshitz (BKL) [7]. The argument is based on formal calculations which originally were without rigorous mathematical proofs. In essence, the BKL picture consists of two predictions: firstly, that the evolution of a spatial point near the singularity is unaffected by the evolution of nearby points. Thus the

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dynamics of an inhomogeneous model can be described by a homogeneous model at each spatial point, with the parameters of the homogeneous model depending on the spatial coordinates. Secondly, the dynamics are expected be oscillatory, similar to the situation in the homogeneous Bianchi IX models. Loosely speaking, the evolution of a spatial point near a cosmological singularity should appear as an infinite sequence of epochs, each similar to a different Kasner spacetime.

Some cosmological models exhibit a simpler behaviour without oscillations, termed 'asymptotically velocity dominated' (AVD) [14, 25]. For these models there is no oscillation and the evolution of a given spatial point approaches a particular Kasner solution. This is exhibited in the evolution equations by the spatial derivative terms becoming negligible in relation to time derivatives.

The behaviour of general spacetimes is still out of reach of the mathematical techniques available, so it makes sense to study particular subclasses of solutions, subject to some particular symmetry and/or a particular choice of matter. Recently there has been considerable progress in the study of spatially homogeneous spacetimes [36, 37, 40, 41, 43]. There is also an interesting result which establishes AVD behaviour for the general Einstein equations coupled to a scalar field, without any symmetry assumptions [1].

It is of obvious interest to relax the symmetry assumptions from homogeneous to inhomogeneous spacetimes. For simplicity, we restrict attention to the spatially compact case in vacuum. The largest symmetry group that is compatible with inhomogeneities is then $U(1) \times U(1)$, which in the spatially compact case leads to the Gowdy models [17, 10]. The possible spatial topologies are then essentially $\mathbf{T}^3$, $\mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{S}^3$. (In the more general case of *local* $U(1) \times U(1)$ symmetry, other topologies are possible, see [42].)

For Gowdy spacetimes it is possible to define a quantity, the 'velocity', which describes the dynamics near the singularity. The terminology is slightly misleading since the velocity is only uniquely defined up to a sign. While a sign can be introduced, it depends on the particular parametrisation of the metric. If AVD holds, the velocity has a (space dependent) limit at the singularity.

For the subclass of polarised Gowdy models, which describe universa with polarised gravitational waves, the Einstein evolution equations reduce to a single linear ordinary differential equation, which can be solved formally in terms of a series of Legendre polynomials. The polarised solutions have been shown to be AVD [12, 25, 20].

For the full Gowdy models with $\mathbf{T}^3$ spatial topology, numerical simulations show that the velocity has a limit between 0 and 1 for almost all spatial points in generic situations [9, 8, 22, 23]. The exceptions are isolated points where the velocity remains outside this range, which results in a discontinuous behaviour, 'spikes', in the limit. A family of singular solutions have been constructed analytically by Kichenassamy and Rendall using Fuchsian methods [28], based on previous work on formal asymptotic expansions by Grubisic and Moncrief [19]. These solutions have a prescribed asymptotic behaviour. In the case when the asymptotic velocity is between 0 and 1, the solutions depend on the full number of free functions, which indicates that they correspond to an open set of initial data. Recently, a similar family of solutions with spikes have been constructed by Rendall and Weaver, showing that the spikes are real features of the model and not numerical artefacts [39].

We should also mention that there is a result on the existence of Gowdy $\mathbf{T}^3$ solutions with small data close to that of a Kasner $(2/3, 2/3, -1/3)$ spacetime [11].

For the unpolarised Gowdy spacetimes with $\mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{S}^3$ topology, much less is known. Garfinkle has done numerical simulations of Gowdy $\mathbf{S}^2 \times \mathbf{S}^1$ spacetimes, which show similar behaviour as in the $\mathbf{T}^3$ case, including the appearance of spikes. It is the purpose of this paper to provide a Fuchsian analysis of Gowdy $\mathbf{S}^2 \times \mathbf{S}^1$ and $\mathbf{S}^3$ spacetimes similar to that in [28] for $\mathbf{T}^3$ spacetimes.

The Fuchsian algorithm [27, 28] has proven to be a valuable tool in showing the existence of singular solutions to nonlinear wave equations. Variants of it has been used to show the existence of Cauchy horizons with certain properties [30], to analyse the properties of isotropic singularities [33, 34, 13, 3, 4, 2], to construct families of singular Gowdy or plane symmetric spacetimes [28, 6, 5, 32, 24] and for the general result on scalar field spacetimes [1].

The original version of the Fuchsian technique only applies to the analytic case, since the proof uses the Cauchy formula in the complex domain to estimate derivatives in terms of functions themselves. There are a number of results extending the Fuchsian algorithm to give smooth solutions [33, 34, 13, 26, 6, 38] in various contexts. In section 4 below we will modify the argument of [38] to apply to our case.

Finally, we note that there is an additional motivation for studying Gowdy spacetimes with $\mathbf{S}^2 \times \mathbf{S}^1$ topology apart from cosmology, since axisymmetric and stationary black hole interiors are in fact Gowdy $\mathbf{S}^2 \times \mathbf{S}^1$ spacetimes. These include the region of Kerr spacetime between the inner and outer horizon [35] and the interiors of the 'distorted black holes' studied by Geroch and Hartle [16].

The outline of the paper is as follows. In section 2 we give an introduction to the Gowdy spacetimes, introduce coordinates and a parametrisation of the metric, and rewrite the field equations in a suitable form. We also discuss the restrictions needed for the metric to be smooth at the axes and the properties of some previously known solutions. In section 3 we give a short review of the Fuchsian technique of [28], and construct analytic solutions of the field equations. This is then extended to smooth solutions in section 4, by means of a generalisation of the technique in [38]. Finally we discuss the shortcomings and implications of the results in section 5.

2 Gowdy spacetimes

We assume that (M, g) is a $U(1) \times U(1)$-symmetric spatially compact spacetime with spacelike group orbits. More precisely, let $M = I \times \Sigma$ where $I \subset \mathbf{R}$ and Σ is a connected orientable compact three manifold, and assume that the $U(1) \times U(1)$ group acts effectively on Σ. It then follows [10] that the topology of Σ is that of either $\mathbf{T}^3$, $\mathbf{S}^2 \times \mathbf{S}^1$, $\mathbf{S}^3$ or a lens space $\mathbf{L}(p, q)$. Since the $\mathbf{T}^3$ case is treated in [28] and [38] and $\mathbf{L}(p, q)$ may be covered by $\mathbf{S}^3$, we only consider $\Sigma \simeq \mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{S}^3$ here.[1]

[1] [28] and [38] treat the $\mathbf{T}^3$ case with vanishing twist. See [24] for a similar treatment with nonvanishing twist but restricted to the polarised case.

We introduce coordinates on Σ as follows. Let θ label the orbits of $U(1) \times U(1)$ in Σ and let (ϕ, χ) be coordinates on each orbit induced by the standard coordinates on $U(1) \times U(1)$. For $\Sigma \simeq \mathbf{S}^2 \times \mathbf{S}^1$, we choose χ to be a cyclic coordinate on the $\mathbf{S}^1$ part, $\chi \in [0, 2\pi] \mod 2\pi$, and (θ, ϕ) to be spherical coordinates on the $\mathbf{S}^2$ part, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi] \mod 2\pi$. This fixes (ϕ, χ) up to translation. For $\Sigma \simeq \mathbf{S}^3$, the fix points to the first $U(1)$ factor of $U(1) \times U(1)$ form a circle S_1, and the set of fix points to the second factor is a similar circle S_2 (this is a nontrivial consequence of the effective action, see, e.g., [31]). The parametrisation is chosen such that S_1 and S_2 correspond to $\theta = 0$ and $\theta = \pi$, respectively. This fixes (ϕ, χ) up to translation in this case as well. Note that the two cases are similar in a neighbourhood of one of the axes.

If (M, g) is assumed to be a maximal globally hyperbolic development of generic data on Σ, then M contains the set $(0, \pi) \times \Sigma$, on which the line element of the metric g may be parametrised as

$$\mathrm{d}s^2 = e^A(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + \mathrm{d}\sigma^2, \tag{1}$$

where $\mathrm{d}\sigma^2$ is a metric on the orbits with determinant $R = c \sin t \sin\theta$ for some positive constant c [10]. We set $c = 1$ from now on, since a constant conformal factor does not affect the qualitative aspects of the model. Here 'generic' refers to an open and dense subset of all Cauchy data; see [10] for the exact definition. We will call this model the Gowdy model over Σ [17].

In the $\mathbf{T}^3$ case, the metric can also be written in the form (1), but with orbit metric determinant t. It follows that for $\mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{S}^3$, the metric can be brought into the $\mathbf{T}^3$ form by a change of coordinates whenever $\theta \neq 0$ and $\theta \neq \pi$. Thus the result of [28] and [38] may be applied locally on any subset of M where θ is bounded away from 0 and π, and what remains is to study a neighbourhood of one of the symmetry axes. (Strictly speaking, in the smooth case it is necessary to show that local solutions can be patched together. This may be done using the domain of dependence result of Theorem 4.1 below.) Also, by the symmetry of the problem, we may restrict attention to a neighbourhood of $\theta = 0$ and $t = 0$. When we write 'the axis' below, we will mean the axis at $\theta = 0$, although all arguments apply to the other axis at $\theta = \pi$ as well.

2.1 Parametrisations

The orbit metric $\mathrm{d}\sigma^2$ may be parametrised in several ways. If we use the parametrisation

$$\mathrm{d}s^2 = e^A\,(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + R\big[e^P(\mathrm{d}\phi + Q\mathrm{d}\chi)^2 + e^{-P}\mathrm{d}\chi^2\big], \tag{2}$$

the Einstein equations decouple as two evolution equations involving P and Q and two constraint equations involving P, Q and A [10]. Let $(t, \theta) \in M \subset \mathbf{R}^2$ with the Minkowski metric $\eta = -\mathrm{d}t^2 + \mathrm{d}\theta^2$ and $(P, Q) \in N \subset \mathbf{R}^2$ with the hyperbolic metric $h = \mathrm{d}P^2 + e^{2P}\mathrm{d}Q^2$. Given the map $\Psi\colon M \ni (t, \theta) \mapsto (P, Q) \in N$, we define a section $\mathrm{d}\Psi$ of $T^*M \otimes \Psi^*(TN)$ by

$$\mathrm{d}\Psi = \mathrm{d}P \otimes \Psi^* \frac{\partial}{\partial P} + \mathrm{d}Q \otimes \Psi^* \frac{\partial}{\partial Q}. \tag{3}$$

Then the evolution equations may be written as the wave-map type equation

$$\operatorname{tr}_\eta\big(\mathcal{D}(h,\eta)(R\,\mathrm{d}\Psi)\big) = 0, \tag{4}$$

where $\mathcal{D}(h,\eta)$ is a connection on $T^*M \otimes \Psi^*(TN)$ defined in terms of the connections of (M,η) and (N,h) (see [10] for details).

The 'velocity' $\nu(t,\theta_0)$ is defined up to a sign as the h-velocity of the solution (P,Q) of (4) along a curve $\theta = \theta_0$, rescaled by an appropriate factor. In the $\mathbf{T}^3$ case, this factor is t, but in our case it is more convenient to take the factor to be $\tan t$. So we define the velocity to be

$$\nu = \sqrt{(DP)^2 + e^{2P}(DQ)^2}, \tag{5}$$

where $D = \tan t\,\partial/\partial t$.

An 'asymptotically velocity dominated' (AVD) solution is a solution which asymptotically approach a solution to the 'AVD equations', obtained by dropping pure spatial derivative terms in the evolution equations [14, 25]. For an AVD solution, it follows immediately from the AVD equations that the velocity ν tends to a finite limit as $t \to 0$. We will refer to this limit as the asymptotic velocity. It is also possible to fix the sign of the velocity to be the same as the sign of DP, note however that this sign depends upon the parametrisation as we will see below.

Note that P cannot be regular at the axis. The regular parametrisation used by Garfinkle in his numerical work [15] is obtained from (2) by setting

$$P = \hat{P} + \ln\sin\theta \qquad \text{and} \qquad A = \hat{P} + \gamma + \ln\sin t, \tag{6}$$

with γ, $\hat{P}$ and Q smooth functions of t and θ. Unfortunately, the corresponding evolution equations contain coefficients that are singular at the axes. This is not a major obstacle for the numerics since the full terms appearing in the equations are well behaved, but it does prevent a direct analytical treatment.

Another parametrisation may be obtained by interchanging the roles of the Killing vectors $\partial/\partial\phi$ and $\partial/\partial\chi$. This corresponds to an inversion in the hyperbolic plane (N,h), so the field equations are invariant under this reparametrisation. Explicitly,

$$\mathrm{d}s^2 = e^A\,(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + R\big[e^Y(\mathrm{d}\chi + X\mathrm{d}\phi)^2 + e^{-Y}\mathrm{d}\phi^2\big], \tag{7}$$

where

$$Y = P + \ln(e^{-2P} + Q^2) \qquad \text{and} \qquad X = \frac{Q}{e^{-2P} + Q^2}. \tag{8}$$

Note that there is a subtlety here. If $P \to \infty$ as $t \to 0$, the rate of blow-up of Y will be dramatically different depending on whether Q vanishes or not. This is a consequence of the parametrisation, and must be distinguished from real geometric features of the model.

The parametrisation (7) is again singular at the axis, but this may be resolved by substituting $Y = Z - \ln R$. We also set $A = \lambda - Z + 2\ln\sin t$. The metric then becomes

$$\mathrm{d}s^2 = e^{\lambda - Z}\sin^2 t\,(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + e^Z(\mathrm{d}\chi + X\mathrm{d}\phi)^2 + R^2 e^{-Z}\mathrm{d}\phi^2. \tag{9}$$

From Lemma 5.1 of [10], (9) defines a smooth metric with the desired symmetry if and only if λ, Z and X are smooth functions of θ on $\mathbf{S}^2$, with X and λ vanishing at $\theta = 0$ and $\theta = \pi$. Note that a function is smooth in a neighbourhood of $\theta = 0$ or $\theta = \pi$ in $\mathbf{S}^2$ if and only if it is a smooth function of $\sin^2\theta$ there.

Because of the invariance of h under inversion, the velocity is

$$\nu = \sqrt{(DY)^2 + e^{2Y}(DX)^2} = \sqrt{(1 - DZ)^2 + e^{2Z}R^{-2}(DX)^2}. \tag{10}$$

As mentioned above, it is possible to assign a sign to ν by setting it equal to the sign of DP. Assume, for illustrative purposes, that Q is independent of t and that $DP \to \infty$ as $t \to 0$. Then it follows from (8) that $DZ \to DP$ as $t \to 0$ wherever $Q \neq 0$, but $DZ \to -DP$ for those values of θ where Q vanishes. So the sign defined with respect to DP cannot be expressed in terms of DZ alone. Also, if DP tends to a smooth function of x as $t \to 0$, the corresponding limit of DZ has discontinuities at points where Q has isolated zeros (note that for any smooth metric of the form (2), Q must vanish at the axes). Since DP has a perfectly regular asymptotic behaviour in this case, these discontinuities are artefacts of the parametrisation (called 'false spikes' in [39]). The same argument may of course also be applied in the other direction, giving discontinuities in the asymptotic behaviour of DP from a well-behaved DZ. For the rest of this paper, we fix the sign of the velocity to be equal to the sign of $1 - DZ$.

2.2 Einstein's equations

In the last section we obtained a parametrisation which is regular at the axis. We will now modify the parametrisation to obtain evolution equations with regular coefficients as well.

We will denote partial derivatives by subscripts, e.g., $f_\theta := \partial_\theta f = \partial f/\partial\theta$. As mentioned above, Einstein's equations decompose into the wave map equation (4) for Z and X and two constraint equations for λ, Z and X. If we denote the covariant derivative of the Minkowski metric η on N by $\mathcal{D}$, (4) can be written as

$$R^{-1}\mathcal{D}_A(R\,\mathcal{D}^A Z) = R^{-2}e^{2Z}\,\mathcal{D}_A X\,\mathcal{D}^A X, \tag{11a}$$

$$R\,\mathcal{D}_A(R^{-1}e^{2Z}\mathcal{D}^A X) = 0, \tag{11b}$$

and the constraint equations are

$$2(\lambda_\pm \pm \cot t)R_\pm = 4(R_\theta)_\pm + R(Z_\pm^2 + e^{2Z}R^{-2}X_\pm^2), \tag{12}$$

where we have used the notation $f_\pm := f_\theta \pm f_t$ [10].

The integrability of the constraints (12) are ensured by the evolution equations (11). When R_+ and R_- are nonzero, the constraints determine λ up to a constant, and the

constant is fixed by the smoothness requirement, i.e., that $\lambda = 0$ at $\theta = 0$. Since $R = \sin t \sin\theta$, R_+ and R_- vanish when $\theta = t$ and $\theta = \pi - t$ respectively. So this happens for two values of θ in each Cauchy surface. At these points, the constraints become 'matching conditions' for Z and X [17, 42]. If the matching conditions hold on an initial Cauchy surface, they are preserved by the evolution equations [15]. In what follows we will assume without further comment that the solutions are chosen such that the matching conditions hold.

To obtain a system with smooth coefficients, we reparametrise the metric using an Ernst potential as in [10], a procedure which is similar to the Kramer-Neugebauer transformation in stationary and axisymmetric spacetimes [29]. Let

$$\Omega := -R^{-1}e^{2Z}(X_\theta\,\mathrm{d}t + X_t\,\mathrm{d}\theta). \tag{13}$$

It then follows from (11b) that Ω is a closed form. Thus we may write $\Omega = \mathrm{d}\omega$ for some function ω, defined up to a constant, on any simply connected open set containing $\theta = 0$. Inverting the relation (13) shows that if ω is a smooth function of θ on a neighbourhood of $\theta = 0$ on $\mathbf{S}^2$, so is X. This determines X up to a constant, which is fixed by the requirement that $X = 0$ at $\theta = 0$ so that the metric is smooth at the axis. Following [39], we will call the transformation $(Y, X) \mapsto (Z, \omega)$ a 'Gowdy-to-Ernst' transformation.

Expressing the evolution equation (11a) in terms of Z and ω gives

$$R^{-1}\mathcal{D}_A(R\,\mathcal{D}^A Z) = -e^{-2Z}\,\mathcal{D}_A\omega\,\mathcal{D}^A\omega, \tag{14a}$$

and the identity $\mathcal{D}_{[A}\mathcal{D}_{B]}X = 0$ together with (13) gives a second evolution equation

$$\mathcal{D}_A(Re^{-2Z}\mathcal{D}^A\omega) = 0. \tag{14b}$$

Expanding (14) we get

$$Z_{tt} + \cot t\, Z_t - Z_{\theta\theta} - \cot\theta\, Z_\theta = -e^{-2Z}(\omega_t^2 - \omega_\theta^2), \tag{15a}$$

$$\omega_{tt} + \cot t\,\omega_t - \omega_{\theta\theta} - \cot\theta\,\omega_\theta = 2(Z_t\,\omega_t - Z_\theta\,\omega_\theta). \tag{15b}$$

We will now rewrite the equations in a form which makes the regularity of the coefficients at the axis explicit, and which is more suited for the Fuchsian techniques. Put $\tau := \sin t$ and $D := \tan t\,\partial_t = \tau\,\partial_\tau$ as before, and let $\Delta := \partial_\theta^2 + \cot\theta\,\partial_\theta + (\sin\theta)^{-2}\partial_\phi^2$, e.g., the Laplacian on $\mathbf{S}^2$ with respect to the metric $\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ induced by the natural embedding in Euclidean $\mathbf{R}^3$. We also write $\nabla f \cdot \nabla g := f_\theta\, g_\theta + (\sin\theta)^{-2} f_\phi\, g_\phi$ and $(\nabla f)^2 := \nabla f \cdot \nabla f$. Since Z and ω are independent of ϕ, (15) can be written as

$$(1-\tau^2)D^2 Z - \tau^2 DZ = \tau^2\Delta Z - e^{-2Z}\big[(1-\tau^2)(D\omega)^2 - \tau^2(\nabla\omega)^2\big], \tag{16a}$$

$$(1-\tau^2)D^2\omega - \tau^2 D\omega = \tau^2\Delta\omega + 2\big[(1-\tau^2)DZ\,D\omega - \tau^2\,\nabla Z\cdot\nabla\omega\big]. \tag{16b}$$

The constraints (12) expressed in terms of Z and ω are

$$D\lambda + \tan^2 t\cot\theta\,\lambda_\theta = -2(1+\tan^2 t) + \frac{1}{2}\Big[(DZ)^2 + \tan^2 t\, Z_\theta^2 + e^{-2Z}\big((D\omega)^2 + \tan^2 t\,\omega_\theta^2\big)\Big], \tag{17a}$$

$$D\lambda + \tan\theta\,\lambda_\theta = \tan\theta\,\big(Z_\theta\, DZ + e^{-2Z}\omega_\theta\, D\omega\big). \tag{17b}$$

In the analytic case, the Fuchsian techniques can in fact be applied directly to (16) in the variables τ and θ. For the smooth case however, we have to cast the equations into symmetric hyperbolic form, which cannot be done using the θ variable alone because the axial symmetry will lead to coefficients singular in θ at $\theta = 0$. We therefore introduce an approximately Cartesian coordinate system on a neighbourhood of $\theta = 0$ on $\mathbf{S}^2$. Let $x := \sin\theta\cos\phi$ and $y := \sin\theta\sin\phi$. In the variables (x, y), the metric on $\mathbf{S}^2$ is

$$\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 = (1 - x^2 - y^2)^{-1}\big[(1 - y^2)\mathrm{d}x^2 + 2xy\,\mathrm{d}x\mathrm{d}y + (1 - x^2)\mathrm{d}y^2\big], \tag{18}$$

hence

$$\Delta = (1 - x^2)\,\partial_x^2 - 2xy\,\partial_x\partial_y + (1 - y^2)\,\partial_y^2 - 2x\,\partial_x - 2y\,\partial_y, \tag{19}$$

and

$$\nabla f \cdot \nabla g = (1 - x^2) f_x g_x - xy(f_x g_y + f_y g_x) + (1 - y^2) f_y g_y. \tag{20}$$

Inserting these expressions in (16) then gives a system in the coordinates t, x and y with coefficients regular in x and y at the axis.

2.3 Restrictions at the axes

As mentioned in the introduction, asymptotic velocity dominance for Gowdy spacetimes may be loosely formulated as that the metric tends to a Kasner metric when $t \to 0$ for a fixed point in Σ. The presence of symmetry axes imposes additional restrictions on the asymptotic behaviour along the axis.

To make this precise we calculate the generalised Kasner exponents. These are defined as the eigenvalues q_i of the renormalised second fundamental form $(\operatorname{tr} k)^{-1} k_{ij}$, expressed in an orthonormal frame on Σ. From the expression for the metric (9) we find that

$$q_1 = (D\lambda - DZ + 2)/(D\lambda - DZ + 4), \tag{21a}$$

$$q_{2,3} = (1 \pm \nu)/(D\lambda - DZ + 4), \tag{21b}$$

where ν is the velocity as given in (10).

Because of the rotational symmetry, the only possible values of the Kasner exponents at the axis are $(0, 0, 1)$ and $(2/3, 2/3, -1/3)$ since the two eigenvalues corresponding to eigenvectors tangent to Σ must agree. From the smoothness requirements $D\lambda$ and DX vanish at the axis, and using our sign convention we have $\nu = 1 - DZ$ there. It then follows immediately from (21) that for the $(0, 0, 1)$ case, $DZ \to 2$ and $\nu \to -1$ as $t \to 0$ along the axis, while for the $(2/3, 2/3, -1/3)$ case, $DZ \to -2$ and $\nu \to 3$.

As noted at the beginning of this section, the $\mathbf{T}^3$ results of [28] applies to sets not containing the axis at $\theta = 0$. In that case, we obtain solutions with the full number of free functions only in the case when $0 < \nu < 1$ (and $DP < 0$) in the limit $t \to 0$. Also, numerical simulations have shown that the velocity is indeed driven into this range, except at isolated spatial points [9] (see also [39] for a heuristic argument). So we cannot hope to show the existence of velocity dominated solutions depending on the full number of free functions since ν is either -1 or 3 on the axis. (Actually, in our case DZ will play the role of ν, but the same conclusion can be drawn since $\nu = 1 - DZ$ on the axis.)

2.4 Some known solutions

2.4.1 Polarised solutions

The polarised solutions form the subclass with $Q \equiv 0$, so called because they describe spacetimes with polarised gravitational waves. In this case the evolution equations (16) reduce to a linear ordinary differential equation for P, the Euler-Poisson-Darboux equation. The solutions may be expressed explicitly in terms of Legendre polynomials [20]. Also, rigorous asymptotic expansions in a neighbourhood of the singularity have been found [25, 12]. In our notation,

$$Z(t,\theta) = k(\theta)\ln\sin t + \varphi(\theta) + u(t,\theta), \tag{22a}$$

$$\lambda(t,\theta) = \frac{1}{2}(k^2-4)\ln\sin t + \psi(\theta) + v(t,\theta), \tag{22b}$$

with $|D^m\partial_\theta^n u| \leq C\sin^2 t|\ln\sin t|$ and $|D^m\partial_\theta^n v| \leq C\sin^2 t|\ln\sin t|^2$ for $m = 0, 1$, all n and a constant C. The functions k and φ may be chosen freely as long as they satisfy the constraints, and the solutions are all asymptotically velocity dominated. Since λ must vanish at the axis for the metric to be smooth, it follows immediately from the expansions that $k = \pm 2$ there, and the velocity $\nu = 1 - k$ must tend to -1 or 3 along the axis. The Kretschmann curvature scalar is unbounded as $t \to 0$ unless $\nu = 1 - k = \pm 1$, $k_\theta = 0$ and $k_{\theta\theta} = 0$.

There is a way in which one may obtained unpolarised solutions from a given polarised solution by means of an Ehlers transformation. This is used to obtain the 'reference solution' used to validate the numerical code in [15] (see also [18]). The technique works because a combination of an Ehlers transformation with Gowdy-to-Ernst transformations gives an isometry of the hyperbolic plane (N, h). The velocity of the transformed solution has the same asymptotic behaviour as for the original polarised solution.

2.4.2 Black hole solutions

The region of Kerr spacetime between the inner and outer horizons may be written as a Gowdy spacetime over $\mathbf{S}^2\times\mathbf{S}^1$, by choosing as time coordinate a rescaling of the usual radial coordinate [35]. In our parametrisation (9), the velocity ν tends to -1 along the axis and to $+1$ elsewhere. The discontinuous behaviour is an artefact of the parametrisation, since in the parametrisation (2), $DP \to -1$ everywhere. This provides an example of Gowdy spacetimes on $\mathbf{S}^2 \times \mathbf{S}^1$ without curvature singularities. The extremal case corresponding to the interior of the Schwarzschild black hole is a polarised Gowdy solution with $\nu \to -1$ at the horizon and $\nu \to 3$ at the curvature singularity.

The same idea can be applied to the 'distorted black hole' spacetimes studied by Geroch and Hartle. These are constructed by analytic continuation of certain Weyl solutions, essentially corresponding to perturbations on an exterior Schwarzschild background, through the horizon. The solutions are static, but they can be different from Schwarzschild spacetime by violating asymptotic flatness or by allowing matter in the exterior region. The interior is a polarised Gowdy spacetime with $\mathbf{S}^2 \times \mathbf{S}^1$ topology. If we let the Schwarzschild

solution be given in Gowdy coordinates with $P = P_S$, the Geroch-Hartle solution is given by $P = P_S + U$, where U is regular on the horizon. It is possible to show that U must be regular at the other singularity as well. Thus the asymptotic velocity is the same as for the interior Schwarzschild solution.

Note that for polarised Gowdy spacetimes, analyticity is a necessary as well as sufficient condition for the existence of an extension with a compact Cauchy horizon [12]. The necessity follows from the fact that on the extension, the evolution equation becomes the Laplace equation in suitably chosen coordinates.

2.4.3 Numerical solutions

Garfinkle has done numerical simulations of Gowdy spacetimes on $\mathbf{S}^2 \times \mathbf{S}^1$, with similar results as in the $\mathbf{T}^3$ case [15]. All solutions have velocity -1 at the axis, without spiky features, which probably can be explained by the particular choice of initial data. We should emphasise however that Garfinkle's calculations are carried out in the parametrisation (6), and rewriting his results in our parametrisation (9) gives solutions with 'false spikes' at the axis. Since we cannot generate solutions with spikes by the method in this paper, our results cannot be compared directly with those in [15].

3 The analytic case

3.1 Fuchsian systems

Consider a system of partial differential equations on $\mathbf{R}^{n+1}$, whose solutions are expected to have a singularity as $t \to 0$. The Fuchsian algorithm is based on the following idea: decompose the unknown into a prescribed singular part, depending on a number of arbitrary functions, and a regular part w. If the system can be rewritten as a Fuchsian system of the form

$$t\partial_t w + N(x)w = tf(t, x, w, w_x), \tag{23}$$

where w_x denotes the collection of spatial derivatives of w, the Fuchsian method applies. In the analytic case we have the following theorem.

Theorem 3.1 (Kichenassamy and Rendall [28]). *Assume that N is an analytic matrix near $x = 0$ such that there is a constant C with $\|\sigma^N\| \leq C$ for $0 < \sigma < 1$, where σ^N is the matrix exponential of $N \ln \sigma$. Also, suppose that f is a locally Lipschitz function of w and w_x which preserves analyticity in x and continuity in t. Then the Fuchsian system (23) has a unique solution in a neighbourhood of $x = 0$ and $t = 0$ which is analytic in x and continuous in t, and tends to 0 as $t \to 0$.*

The method of proof is a variation of the standard Cauchy problem. Note that Theorem 3.1 also applies to the case

$$t\partial_t w + N(x)w = t^\alpha f(t, x, w, w_x), \tag{24}$$

since changing the t variable to t^α transforms the system into the form (23).

3.2 Low velocity, analytic case

Let k, φ, ω_0 and ψ be real analytic functions of θ on a neighbourhood of $\theta = 0$ in $\mathbf{S}^2$, with $k \in (0,1)$. We introduce new unknowns u and v such that

$$Z(\tau,\theta) = k(\theta)\ln\tau + \varphi(\theta) + \tau^{\epsilon}u(\tau,\theta), \tag{25a}$$
$$\omega(\tau,\theta) = \omega_0(\theta) + \tau^{2k(\theta)}\big(\psi(\theta) + v(\tau,\theta)\big), \tag{25b}$$

where $\epsilon > 0$ is a small constant which we will fix later. The idea is to use Theorem 3.1 to show that u, v, Du and Dv tend to 0 as $t \to 0$, so that (25) provides an asymptotic expansion of Z and ω. The term 'low velocity' is then justified by the fact that the velocity ν tends to $1-k \in (0,1)$ as $t \to 0$, as is easily verified from the definition (10). For this reason we will refer to $1-k$ as the 'asymptotic velocity'. However, remember that from the discussion in section 2.3, the velocity should be -1 or 3 at the axis, so the solutions constructed in this section are *not* regular at the axis.

In terms of X, the expansion (25b) of ω corresponds to

$$X = X_0(\theta) + \tau^{2(1-k(\theta))}\big(\tilde\psi(\theta) + \tilde v(\tau,\theta)\big), \tag{26}$$

for some functions X_0, $\tilde\psi$ and $\tilde v$, which can be compared directly with the $\mathbf{T}^3$ results [28].

Inserting the expansion (25) into the system (16) gives

$$\begin{aligned}(1-\tau^2)D^2u = &-(1-\tau^2)\big[2\epsilon Du + \epsilon^2 u\big] + \tau^{2-\epsilon}\big[k + (\ln\tau)\Delta k + \Delta\varphi\big] + \tau^2\big[(D+\epsilon)u + \Delta u\big]\\ &- e^{-2\varphi-2\tau^{\epsilon}u}\Big\{(1-\tau^2)\tau^{2k-\epsilon}\big[(D+2k)(\psi+v)\big]^2 - \tau^{2-2k-\epsilon}(\nabla\omega_0)^2\\ &\qquad - 2\tau^{2-\epsilon}\nabla\omega_0\cdot\big[\nabla(\psi+v) + 2(\ln\tau)(\psi+v)\nabla k\big]\\ &\qquad - \tau^{2+2k-\epsilon}\big[\nabla(\psi+v) + 2(\ln\tau)(\psi+v)\nabla k\big]^2\Big\},\end{aligned} \tag{27a}$$

$$\begin{aligned}(1-\tau^2)D^2v = &-2k(1-\tau^2)Dv + 2(1-\tau^2)\tau^{\epsilon}(D+\epsilon)u(D+2k)(\psi+v)\\ &+ \tau^2\big[2(\ln\tau)\nabla k\cdot\nabla(\psi+v) + 2(\ln\tau)(\psi+v)\Delta k\\ &\qquad + (D+2k)(\psi+v) + \Delta(\psi+v)\big]\\ &+ \tau^{2-2k}\Delta\omega_0 - 2\tau^{2-2k}\nabla\omega_0\cdot\big[(\ln\tau)\nabla k + \nabla\varphi + \tau^{\epsilon}\nabla u\big]\\ &- 2\tau^2\big[\nabla\varphi + \tau^{\epsilon}\nabla u\big]\cdot\big[2(\ln\tau)(\psi+v)\nabla k + \nabla(\psi+v)\big].\end{aligned} \tag{27b}$$

Next, we introduce the variables

$$U_0 := u, \qquad U_1 := Du, \qquad U_2 := \tau u_x, \qquad U_3 := \tau u_y, \tag{28a}$$
$$V_0 := v, \qquad V_1 := Dv, \qquad V_2 := \tau v_x, \qquad V_3 := \tau v_y. \tag{28b}$$

We also write $\overline{U}$ and $\overline{V}$ as shorthands for the vectors (U_2, U_3) and (V_2, V_3). Using (19) and

(20) the system (27) may then be written as

$$
\begin{aligned}
DU_0 &= U_1, && \text{(29a)}\\
(1-\tau^2)DU_1 &= -(1-\tau^2)\big[2\epsilon U_1+\epsilon^2 U_0\big]+\tau^{2-\epsilon}\big[k+(\ln\tau)\Delta k+\Delta\varphi\big]+\tau^2(U_1+\epsilon U_0)\\
&\quad +\tau\big[(1-x^2)U_{2x}-xy(U_{2y}+U_{3x})+(1-y^2)U_{3y}-2xU_2-2yU_3\big]\\
&\quad -e^{-2\varphi-2\tau^\epsilon U_0}\Big\{(1-\tau^2)\tau^{2k-\epsilon}\big[V_1+2k(\psi+V_0)\big]^2-\tau^{2-2k-\epsilon}(\nabla\omega_0)^2 && \text{(29b)}\\
&\qquad -2\tau^{1-\epsilon}\nabla\omega_0\cdot\big[\overline{V}+\tau\nabla\psi+2\tau(\ln\tau)(\psi+V_0)\nabla k\big]\\
&\qquad -\tau^{2k-\epsilon}\big[\overline{V}+\tau\nabla\psi+2\tau(\ln\tau)(\psi+V_0)\nabla k\big]^2\Big\},\\
DU_2 &= \tau(U_{0x}+U_{1x}), && \text{(29c)}\\
DU_3 &= \tau(U_{0y}+U_{1y}), && \text{(29d)}\\
DV_0 &= V_1, && \text{(29e)}\\
(1-\tau^2)DV_1 &= -2k(1-\tau^2)V_1+2(1-\tau^2)\tau^\epsilon(U_1+\epsilon U_0)\big(V_1+2k(\psi+V_0)\big)\\
&\quad +\tau^2\big[V_1+2k(\psi+V_0)+2(\ln\tau)(\psi+V_0)\Delta k+2(\ln\tau)\nabla k\cdot\nabla\psi+\Delta\psi\big]\\
&\quad +2\tau(\ln\tau)\nabla k\cdot\overline{V}+\tau^{2-2k}\Delta\omega_0\\
&\quad +\tau\big[(1-x^2)V_{2x}-xy(V_{2y}+V_{3x})+(1-y^2)V_{3y}-2xV_2-2yV_3\big] && \text{(29f)}\\
&\quad -2\tau^{2-2k}\nabla\omega_0\cdot\big[(\ln\tau)\nabla k+\nabla\varphi+\tau^\epsilon\nabla U_0\big]\\
&\quad -2\tau\big[\tau\nabla\varphi+\tau^\epsilon\overline{U}\big]\cdot\big[2(\ln\tau)(\psi+V_0)\nabla k+\nabla(\psi+V_0)\big],\\
DV_2 &= \tau(V_{0x}+V_{1x}), && \text{(29g)}\\
DV_3 &= \tau(V_{0y}+V_{1y}). && \text{(29h)}
\end{aligned}
$$

If we choose ϵ such that $0<\epsilon<\min\{2k,2-2k\}$, the system (29) is of the form (24) for some α (after dividing (29b) and (29f) by $1-\tau^2$), and a rescaling of τ gives a Fuchsian system of the form (23). By the criterion in [1], or by direct computation of σ^N, the conditions of Theorem 3.1 are fulfilled. (Note that the essential features of the system, in particular the matrix N and the powers of τ, are the same as in the $\mathbf{T}^3$ case [28].) We come to the following conclusion.

Theorem 3.2. *Suppose that k, φ, ω_0 and ψ are real analytic functions of θ in a neighbourhood of $\theta=0$ in $\mathbf{S}^2$ such that $k\in(0,1)$ for each θ. Let ϵ be a positive constant less than $\min\{2k,2-2k\}$. There exists a unique solution of Einstein's equations* (11) *of the form* (25) *in a neighbourhood of $\theta=0$ and $t=0$ such that u, v, Du and Dv all tend to* 0 *as $t\to 0$.*

3.3 Negative velocity, analytic case

In the case when we only assume k to be positive, allowing values of k greater than 1, we may use the same expansion (25) as in the low velocity case. However, examining the system (29) we see that we must assume that ω_0 is constant in order to avoid negative

powers of τ. This is again similar to the $\mathbf{T}^3$ case, and is to be expected since numerical simulations indicate that spiky features may appear as $t \to 0$ [28, 15]. Again, using the definition (10) of the velocity shows that the asymptotic velocity is $1-k$, which is negative if $k > 1$. The expansion (25b) of ω implies

$$X = X_0(\theta) + \tau^{\epsilon}\tilde{v}(\tau, \theta). \tag{30}$$

for some X_0 and $\tilde{v}$, which is in agreement with [28].

By the same arguments as in the low velocity case we have a similar existence theorem for a smaller set of data, including the regular case with velocity -1 at the axis.

Theorem 3.3. *Suppose that k, φ and ψ are real analytic functions of θ in a neighbourhood of $\theta = 0$ in $\mathbf{S}^2$ such that $k > 0$ for each θ, and suppose that ω_0 is a constant. Let ϵ be a positive constant less than $2k$. There exists a unique solution of Einstein's equations* (11) *of the form* (25) *in a neighbourhood of $\theta = 0$ and $t = 0$ such that u, v, Du and Dv all tend to* 0 *as $t \to 0$.*

3.4 High velocity, analytic case

If k is negative, we replace the expansion (25) with

$$Z(\tau, \theta) = k(\theta)\ln\tau + \varphi(\theta) + \tau^{\epsilon}u(\tau, \theta), \tag{31a}$$
$$\omega(\tau, \theta) = \omega_0(\theta) + \tau^{\epsilon}v(\tau, \theta). \tag{31b}$$

Note that we do not get the full number of free functions in this case either. Again, this is because of the possibility of spiky features in the asymptotic behaviour. The calculations are similar to the $k > 0$ case and will not be repeated here. It turns out that the system is valid for $k < 1/2$ (corresponding to an asymptotic velocity greater than $1/2$), by choosing $\max\{0, 2k\} < \epsilon < \min\{2, 2-2k\}$. The corresponding expansion of X is

$$X = X_0 + \tau^{2(1-k(\theta))}(\tilde{\psi}(\theta) + \tilde{v}(\tau, \theta)),$$

for some functions $\tilde{\psi}$ and $\tilde{v}$ and an integration constant X_0 (which must vanish for the solution to be smooth at the axis). These solutions include the regular case with velocity 3 at the axis.

Theorem 3.4. *Suppose that k, φ and ω_0 are real analytic functions of θ in a neighbourhood of $\theta = 0$ in $\mathbf{S}^2$, such that $k < 1/2$ for each θ. Let ϵ be a positive constant such that* $\max\{0, 2k\} < \epsilon < \min\{2, 2-2k\}$. *There exists a unique solution of Einstein's equations* (11) *of the form* (31) *in a neighbourhood of $\theta = 0$ and $t = 0$ such that u, v, Du and Dv all tend to* 0 *as $t \to 0$.*

4 The smooth case

In the previous section we established existence of real analytic solutions with the desired asymptotic behaviour in a number of cases. Here we will consider the corresponding smooth solutions, using a generalisation of the approximation scheme in [38].

The basic idea is to approximate smooth asymptotic data $(k, \varphi, \omega_0, \psi)$ by a sequence of analytic data $(k_m, \varphi_m, \omega_{0m}, \psi_m)$, and show convergence of the corresponding analytic solutions w_m to a smooth solution w. Since the argument does not depend on the details of the Gowdy models, we will study a more general symmetric hyperbolic system with a singular term and sufficiently well-behaved coefficients. Also, since we are only interested in a neighbourhood of one of the axes in $\mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{S}^3$, we will carry out the argument on a subset of $[0, \infty) \times \mathbf{R}^n$.

Definition 4.1. We say that the system of differential equations

$$tA^0(t,x)\partial_t w + N(x)w + tA^j(t,x,w)\partial_j w = tf(t,x,w) \tag{32}$$

is *regular symmetric hyperbolic* if A^0 is uniformly positive definite and symmetric, the A^j are symmetric, and all coefficients are assumed to be regular in a sense to be specified below.

Note that our terminology differs from the usual one here, since the full system (32) is not regular at $t = 0$ in the usual sense. In [38], the case when A^0 is the identity was considered. We will therefore only give an outline of how to generalise the argument to the case of more general A^0.

In our case, the system (32) can be written as the Fuchsian system

$$t\partial_t w + \tilde{N}(x)w = t\tilde{f}(t,x,w,w_x). \tag{33}$$

Indeed, if A^0 were independent of t, we could put $\tilde{N} := (A^0)^{-1}N$ and $\tilde{f} := (A^0)^{-1}(f - A^j\partial_j w)$. Now the t-dependence of A^0 in our case is such that we may write $(A^0)^{-1}(t,x) = B_0(x) + t^\alpha B_1(t,x)$ for some regular matrices B_0 and B_1 and a constant α for small enough t. The t dependence of $(A^0)^{-1}N$ may then be included in $\tilde{f}$ (after rescaling t if $\alpha < 1$). The techniques of [38] can then be applied to (33) to obtain formal solutions (see below).

Note also that most of the arguments below hold in the more general case when A^0 depends on w as well. However, in that case it is often not possible to rewrite the symmetric hyperbolic system as a Fuchsian system of the form (33).

4.1 Regularity and formal solutions

Here we recall some results from [38] on the existence of formal solutions. We first need appropriate notions of regularity and of formal solutions.

Definition 4.2. A function $f(t,x)$ from an open subset $\Omega \subset [0,\infty) \times \mathbf{R}^n$ to $\mathbf{R}^m$ is called *regular* if it is C^∞ for all $t > 0$ and if its partial derivatives of any order with respect to $x \in \mathbf{R}^n$ extend continuously to $t = 0$ (within $\overline{\Omega}$).

Definition 4.3. A finite sequence $(w_1, w_2, \ldots, w_p)$ of functions defined on an open subset $\Omega \subset [0, \infty) \times \mathbf{R}^n$ is called a *formal solution of order* p of the differential equation (33) on Ω if

1. each w_i is regular and
2. $t\partial_t w_i + \tilde{N}(x) w_i - t\tilde{f}(t, x, w_i, w_{ix}) = \mathcal{O}(t^i)$ for all i as $t \to 0$ in Ω.

The existence of formal solutions to a system of the form (33) is provided by the following result.

Lemma 4.1 (Rendall [38]). *If $\tilde{f}$ is regular and $\tilde{N}$ is smooth and satisfies $\|\sigma^{\tilde{N}}\| \leq C$ for some constant C and all σ in a neighbourhood of* 0*, then* (33) *has a formal solution of any given order which vanishes at $t = 0$.*

By the discussion in the previous section, a formal solution of (33) may also be regarded as a formal solution of (32) with a definition similar to Definition 4.3. We have thus established the existence of formal solutions to (32) of any order.

In proving the existence of smooth solutions to (32) it is important that the matrix N is positive definite. This is not the case in general, so we need to modify the system to fulfil this requirement. The idea is to use a formal solution $\{w_1, \ldots, w_i\}$ and to study the system satisfied by $z_i := t^{1-i}(w - w_i)$, where w is the sought solution of the original system (32). The procedure is similar to that in [38], so the details will be omitted. We end up with a system

$$tA^0(t, x)\partial_t z_i + \big(N(x) + (i-1)A^0(t, x)\big) z_i + tA^j(t, x, w_i + t^{i-1} z_i)\partial_j z_i = t f_i(t, x, z_i), \quad (34)$$

where the regular function f_i is constructed from A^j and f and depend on w_i and w_{ix} as well. The point is that since A^0 is uniformly positive definite, the coefficient $N + (i-1)A^0$ is positive definite for large enough i. We will refer to (34) as the positive definite system.

4.2 The existence theorem

Given smooth asymptotic data $S := \{k, \varphi, \omega_0, \psi\}$ on $\mathcal{U} \subset \mathbf{R}^n$, we may construct a sequence of analytic data $S_m := \{k_m, \varphi_m, \omega_{0m}, \psi_m\}$ on $\mathcal{U}$ which converges to S in $C^\infty(\mathcal{U})$, uniformly on compact subsets. If the formal solutions are constructed as in the proof of Lemma 4.1 (see [38]), the analytic formal solutions w_{mi} of order i corresponding to the analytic data S_m converge to a formal solution w_i of order i corresponding to the smooth data S as $m \to \infty$, and the convergence is uniform on compact subsets. This also holds for spatial derivatives of any order. Hence spatial derivatives of any order of the coefficients of the positive definite system (34) converge on compact subsets as $m \to \infty$. It follows that on any compact subset there is an i such that the coefficient involving $N(x)$ is positive definite for all m in our case, so we fix such a value of i and omit the index i from now on.

The global existence theorem for Gowdy spacetimes over $\mathbf{S}^2 \times \mathbf{S}^1$ and $\mathbf{S}^3$ [10] implies that there are smooth solutions of (34) on a common time interval for all m. Thus our problem can be solved by proving the following theorem.

Theorem 4.1. *Let $z_m(t,x)$ be a sequence of regular solutions on $[0,t_1)\times\mathcal{U}\subset[0,\infty)\times\mathbf{R}^n$, with $z_m(0,x)=0$, to a sequence of regular symmetric hyperbolic equations*

$$tA^0_m(t,x)\partial_t z_m + N_m(x)z_m + tA^j_m(t,x,z_m)\partial_j z_m = tf_m(t,x,z_m). \tag{35}$$

Suppose that N_m is positive definite for each m and that the coefficients converge uniformly to A^0_0, N_0, A^j_0 and f_0 on compact subsets as $m\to 0$, with the same properties as A_m, N_m, A^j_m and f_m, and that the corresponding spatial derivatives converge uniformly as well. Then z_m converges to a regular solution z_0 of the corresponding system with coefficients A^0_0, N_0, A^j_0 and f_0 on $[0,t_0)\times\mathcal{U}$ for some t_0, and $z_0(0,x)=0$.

Proof. The idea is to use energy estimates to show that $\{z_m\}$ is a Cauchy sequence. Since the proof is very similar to the case when A^0 is the identity [38], we only give an outline of the important steps here, with emphasis on the differences.

First we consider the system satisfied by spatial derivatives of z. The collection $\mathbf{z}$ of spatial derivatives up to order s satisfies a system similar to (35),

$$t\mathbf{A}^0_m(t,x)\partial_t\mathbf{z}_m + \mathbf{N}_m(x)\mathbf{z}_m + t\mathbf{A}^j_m(t,x,z_m)\partial_j\mathbf{z}_m = t\,\mathbf{f}_m(t,x,z_m), \tag{36}$$

obtained by differentiating (35) and substituting the equations for lower order spatial derivatives. The problem is that $\mathbf{N}_m$ is not necessarily positive definite, due to the presence of off-diagonal blocks depending on $(A^0_m)^{-1}$ and spatial derivatives of A^0_m and N_m. But this can be dealt with by multiplying the spatial derivatives $D^\alpha z_m$ by $K^{|\alpha|}$ for a sufficiently small constant K as in [38] (here α is a multi-index). Let $\mathbf{z}_m := \{K^{|\alpha|}D^\alpha z_m; |\alpha|\le s\}$ be the collection of weighted spatial derivatives up to order s and assume that $\mathbf{A}_m$, $\mathbf{N}_m$, $\mathbf{A}^j_m$ and $\mathbf{f}_m$ are the corresponding weighted coefficient matrices. For example, in one spatial dimension the system for $\mathbf{z}_m=(z_m,\partial_x z_m)$ has coefficients

$$\mathbf{A}^0_m=\begin{bmatrix}A^0_m & 0\\ 0 & A^0_m\end{bmatrix},\ \mathbf{N}_m=\begin{bmatrix}N_m & 0\\ KD_mN_m & N_m\end{bmatrix},\ \mathbf{A}^j_m=\begin{bmatrix}A^j_m & 0\\ KD_mA^j_m & A^j_m\end{bmatrix},\ \mathbf{f}_m=\begin{bmatrix}f_m\\ KD_mf_m\end{bmatrix}, \tag{37}$$

where D_m is the operator $\partial_x-(\partial_x A^0_m)(A^0_m)^{-1}$. The convergence of the coefficients for any fixed s follows from the convergence of the coefficients of the original equation.

We will need appropriately weighted Sobolev norms

$$\|u\|_{H^s} := (\sum_{|\alpha|\le s} K^{2|\alpha|}\langle D^\alpha u, A^0_m D^\alpha v\rangle_{L^2})^{1/2}. \tag{38}$$

Note that these are equivalent to the usual Sobolev norms. The norms depend a priori on m since they include a factor A^0_m, but since A^0_m converges uniformly on compact subsets to A^0 which is uniformly positive definite, the equivalence can be taken to be independent of m on compact subsets for sufficiently large m.

Second, a domain of dependence result may be obtained by standard techniques. The argument is the same as that in [38] and will be omitted. Thus we need only consider the problem on a compact subset.

Next, we will show that the sequence $\|z_m\|_{H^s}$ is bounded. Differentiating and using the definitions of $\mathbf{z}_m$ and $\mathbf{A}^0_m$ we get

$$\partial_t(\|z_m\|^2_{H^s}) = 2\langle \mathbf{A}^0_m \partial_t \mathbf{z}_m, \mathbf{z}_m\rangle_{L^2} + \sum_{|\alpha|\le s} K^{2|\alpha|}\langle D^\alpha z_m, (\partial_t A^0_m) D^\alpha z_m\rangle_{L^2}, \tag{39}$$

and using (36) to substitute for $\mathbf{A}^0_m \partial_t \mathbf{z}_m$ gives

$$\partial_t(\|z_m\|^2_{H^s}) = -2t^{-1}\langle \mathbf{N}_m \mathbf{z}_m, \mathbf{z}_m\rangle_{L^2} + R_m, \tag{40}$$

where R_m contains the same terms as in the regular case with $\mathbf{N}_m \equiv 0$. The first term on the right is negative since $\mathbf{N}_m$ is positive definite by construction, and R_m may be estimated as in the regular case, giving

$$R_m \le C_1\|z_m\|^2_{H^s} + C_2\|z_m\|_{H^s}. \tag{41}$$

Here C_1 and C_2 are polynomials in $\|(A^0)^{-1}\|_{L^\infty}$, $\|D^\alpha A^0\|_{L^\infty}$, $\|D^\alpha A^j\|_{L^\infty}$ and $\|D^\alpha f\|_{L^\infty}$ for $|\alpha| \le s$, and in $\|z_m\|_{H^k}$ for any given $k > n/2+1$ (by the Sobolev embedding theorem). Since all of the coefficients converge and we may choose $k = s$ if $s > n/2+1$, $\partial_t(\|z_m\|^2_{H^s})$ is bounded by a polynomial in $\|z_m\|_{H^s}$ whose coefficients are independent of m. Applying Gronwalls lemma then gives that $\|z_m\|_{H^s}$ is bounded.

We can in fact obtain a stronger estimate. Since $\|z_m\|_{H^s}$ is bounded, it follows that the coefficients C_1 and C_2 are bounded, so they can be chosen to be constants. A second application of Gronwalls lemma then gives that $t^{-1}\|z_m\|_{H^s}$ is bounded if $s > n/2+1$.

Finally, we show that $\{z_m\}$ is a Cauchy sequence in the H^s norm. Let the difference between consecutive elements of the sequence be given by $v_m := z_m - z_{m-1}$ and put $\mathbf{v}_m := \mathbf{z}_m - \mathbf{z}_{m-1}$. From (36) we get

$$\begin{aligned}\tfrac{1}{2}\,\partial_t(\|v_m\|^2_{H^s}) = &- t^{-1}\langle \mathbf{N}_m \mathbf{v}_m, \mathbf{v}_m\rangle_{L^2} - t^{-1}\langle (\mathbf{N}_m - \mathbf{N}_{m-1})\mathbf{z}_{m-1}, \mathbf{v}_m\rangle_{L^2} \\ &- \langle (\mathbf{A}^0_m - \mathbf{A}^0_{m-1})\partial_t \mathbf{z}_{m-1}, \mathbf{v}_m\rangle_{L^2} - \langle \mathbf{A}^j_m(z_m)\partial_j \mathbf{v}_m, \mathbf{v}_m\rangle_{L^2} \\ &+ \langle \mathbf{v}_m, \mathbf{g}_m\rangle_{L^2},\end{aligned} \tag{42}$$

where

$$\mathbf{g}_m := t\big(\mathbf{f}_m(z_m) - \mathbf{f}_{m-1}(z_{m-1})\big) - t\big(\mathbf{A}^j_m(z_m) - \mathbf{A}^j_{m-1}(z_{m-1})\big)\partial_j \mathbf{z}_{m-1}. \tag{43}$$

The first term on the right hand side of (42) is negative and can be discarded, and using that $t^{-1}\|z_{m-1}\|_{H^s}$ is bounded, the second term is less than $C\|\mathbf{N}_{m+1} - \mathbf{N}_m\|_{L^\infty}\|v_m\|_{H^s}$. To estimate the third term we need a bound on $\|\partial_t z_{m-1}\|_{H^s}$, but this can be obtained by applying the bound on $t^{-1}\|z_{m-1}\|_{H^{s+1}}$ to (36). The fourth term may be estimated by doing a partial integration as in the regular case. The last term is estimated by inserting some terms and applying the mean value theorem and the bound on $\|z_{m-1}\|_{H^{s+1}}$. We end up with an estimate of the form

$$\partial_t(\|v_m\|^2_{H^s}) \le C_1\|v_m\|^2_{H^s} + C_2\|v_m\|_{H^s}, \tag{44}$$

where C_1 is a constant independent of m and C_2 is a polynomial in $\|\mathbf{A}_m^0 - \mathbf{A}_{m-1}^0\|_{L^\infty}$, $\|\mathbf{N}_m - \mathbf{N}_{m-1}\|_{L^\infty}$, $\|\mathbf{A}_m^j - \mathbf{A}_{m-1}^j\|_{L^\infty}$ and $\|\mathbf{f}_m - \mathbf{f}_{m-1}\|_{L^\infty}$ which tends to 0 as $m \to \infty$. Applying Gronwalls Lemma again, we conclude that $\|v_m\|_{H^s} \to 0$ as $m \to \infty$.

We have shown that the solutions z_m of the positive definite system (35) converges in the H^s norm to a smooth solution for each s. A potential problem is that the time interval of convergence may depend on s. But by standard techniques for symmetric hyperbolic equations the solutions may be extended to a common time interval, so we can conclude that z_m converges as smooth functions as well. $\square$

4.3 Low velocity, smooth case

From the discussions in the previous section it follows that we only need to write the Gowdy equations (29) in symmetric hyperbolic form, since the existence of a smooth solution is then guaranteed by Theorem 4.1. We rewrite (29) as

$$DU_0 = U_1, \tag{45a}$$

$$\begin{aligned}(1-\tau^2)DU_1 = &-(1-\tau^2)\big[2\epsilon U_1 + \epsilon^2 U_0\big] + \tau^{2-\epsilon}\big[k + (\ln\tau)\Delta k + \Delta\varphi\big] + \tau^2(U_1 + \epsilon U_0)\\ &+ \tau\big[(1-x^2)U_{2x} - xy(U_{2y} + U_{3x}) + (1-y^2)U_{3y} - 2xU_2 - 2yU_3\big]\\ &- e^{-2\varphi - 2\tau^\epsilon U_0}\Big\{(1-\tau^2)\tau^{2k-\epsilon}\big[V_1 + 2k(\psi + V_0)\big]^2 - \tau^{2-2k-\epsilon}(\nabla\omega_0)^2\\ &\qquad - 2\tau^{1-\epsilon}\nabla\omega_0\cdot\big[\overline{V} + \tau\nabla\psi + 2\tau(\ln\tau)(\psi + V_0)\nabla k\big]\\ &\qquad - \tau^{2k-\epsilon}\big[\overline{V} + \tau\nabla\psi + 2\tau(\ln\tau)(\psi + V_0)\nabla k\big]^2\Big\},\end{aligned} \tag{45b}$$

$$(1-x^2)DU_2 - xyDU_3 = (1-x^2)U_2 - xyU_3 + \tau(1-x^2)U_{1x} - \tau xyU_{1y}, \tag{45c}$$

$$(1-y^2)DU_3 - xyDU_2 = (1-y^2)U_3 - xyU_2 + \tau(1-y^2)U_{1y} - \tau xyU_{1x}, \tag{45d}$$

$$DV_0 = V_1, \tag{45e}$$

$$\begin{aligned}(1-\tau^2)DV_1 = &-2k(1-\tau^2)V_1 + 2(1-\tau^2)\tau^\epsilon(U_1 + \epsilon U_0)\big(V_1 + 2k(\psi + V_0)\big)\\ &+ \tau^2\big[V_1 + 2k(\psi + V_0) + 2(\ln\tau)(\psi + V_0)\Delta k + 2(\ln\tau)\nabla k\cdot\nabla\psi + \Delta\psi\big]\\ &+ 2\tau(\ln\tau)\nabla k\cdot\overline{V} + \tau^{2-2k}\Delta\omega_0\\ &+ \tau\big[(1-x^2)V_{2x} - xy(V_{2y} + V_{3x}) + (1-y^2)V_{3y} - 2xV_2 - 2yV_3\big]\\ &- 2\tau^{2-2k}\nabla\omega_0\cdot\big[(\ln\tau)\nabla k + \nabla\varphi\big] - 2\tau^{1+\epsilon-2k}\nabla\omega_0\cdot\overline{U}\\ &- 2\tau^\epsilon\big[\tau^{1-\epsilon}\nabla\varphi + \overline{U}\big]\cdot\big[2\tau(\ln\tau)(\psi + V_0)\nabla k + \tau\nabla\psi + \overline{V}\big],\end{aligned} \tag{45f}$$

$$(1-x^2)DV_2 - xyDV_3 = (1-x^2)V_2 - xyV_3 + \tau(1-x^2)V_{1x} - \tau xyV_{1y}, \tag{45g}$$

$$(1-y^2)DV_3 - xyDV_2 = (1-y^2)V_3 - xyV_2 + \tau(1-y^2)V_{1y} - \tau xyV_{1x}. \tag{45h}$$

Because of the factor $\tau^{1+\epsilon-2k}$, we must restrict k to the case $0 < k < 3/4$. After an appropriate rescaling of the time variable, the system is clearly a regular symmetric hyperbolic system of the form (32). Forming the corresponding positive definite system (where $i = 3$ suffices to make the coefficient of the singular term positive definite) and applying Theorem 4.1 then gives the existence of the desired solutions.

Note that there is a slight difficulty in showing that (45) is equivalent to (29). For example, from (45c), (45d) and (45a) it follows that $D(U_2 - \tau\partial_x U_0) = U_2 - \tau\partial_x U_0$, and since U_2 and U_0 vanish at $\tau = 0$ we must have $U_2 - \tau\partial_x U_0 = c(x)\tau$ for some function $c(x)$. But there seems to be no direct way of showing that $c(x) \equiv 0$ follows from (45). This must be the case however since we obtain the solution as the limit of a sequence of analytic solutions with this property.

We summarise this section in the following theorem.

Theorem 4.2. *If k, φ, ω_0 and ψ are smooth functions of θ and $0 < k < 3/4$ for all θ, there exists a solution of Einstein's equations* (11) *in a neighbourhood $\mathcal{U}$ of $\theta = 0$ of the form* (25) *where $2k - 1 < \epsilon < \min\{2k, 2 - 2k\}$ and u and v are regular and tend to 0 as $t \to 0$. Given the form of the expansion and a choice of ϵ, the solution is unique.*

4.4 Negative velocity, smooth case

Since exponents of τ involving $-2k$ only appear in terms containing $\nabla\omega_0$, the argument in the previous section applies immediately to the negative velocity case with $k > 0$ and constant ω_0.

Theorem 4.3. *If k, φ and ψ are smooth functions of θ such that $k > 0$ for all θ and ω_0 is a constant, there exists a solution of Einstein's equations* (11) *in a neighbourhood $\mathcal{U}$ of $\theta = 0$ of the form* (25) *where $\epsilon < 2k$ and u and v are regular and tend to 0 as $t \to 0$. Given the form of the expansion and a choice of ϵ, the solution is unique.*

4.5 High velocity, smooth case

The calculations for the high velocity case are similar to the low velocity case and will be omitted. We have the following existence result.

Theorem 4.4. *If k, φ and ω_0 are smooth functions of θ and $k < 1/2$ for all θ, there exists a solution of Einstein's equations* (11) *in a neighbourhood $\mathcal{U}$ of $\theta = 0$ of the form* (31) *where $\max\{0, 2k\} < \epsilon < \min\{2, 2 - 2k\}$ and u and v are regular and tend to 0 as $t \to 0$. Given the form of the expansion and a choice of ϵ, the solution is unique.*

4.6 Intermediate velocity, smooth case

It remains to treat the case when $3/4 \leq k < 1$. The idea is to include one more term in the expansion (25a) of Z,

$$Z(\tau, \theta) = k(\theta) \ln \tau + \varphi(\theta) + \alpha(\theta)\tau^{2-2k} + \tau^{2-2k+\epsilon} u(\tau, \theta), \tag{46}$$

where α is to be chosen such as to eliminate the leading order terms in the equations. We keep the original expansion for ω as in (25b).

The resulting system is similar to (45), but with DU_1 and DU_2 given by

$$
\begin{aligned}
(1-\tau^2)DU_1 =& -(1-\tau^2)\big[(4-4k+2\epsilon)U_1+(2-2k+\epsilon)^2U_0\big]-4\tau(\ln\tau)\nabla k\cdot\overline{U}\\
&+\tau^{2k-\epsilon}\big[k+(\ln\tau)\Delta k+\Delta\varphi\big]+\tau^{2-\epsilon}(4k^2-10k+6)\alpha\\
&+\tau^{2-\epsilon}\big[\Delta\alpha-2(\ln\tau)(\alpha\Delta k+2\nabla\alpha\cdot\nabla k)+4(\ln\tau)^2\alpha(\nabla k)^2\big]\\
&+\tau^2\big[U_1+(2-2k+\epsilon)U_0-2(\ln\tau)U_0\Delta k+4(\ln\tau)^2U_0(\nabla k)^2\big]\\
&+\tau\big[(1-x^2)U_{2x}-xy(U_{2y}+U_{3x})+(1-y^2)U_{3y}-2xU_2-2yU_3\big]\\
&-\exp(-2\varphi-2\alpha\tau^{2-2k}-2\tau^{2-2k+\epsilon}U_0) \qquad\qquad (47a)\\
&\quad\times\Big\{(1-\tau^2)\tau^{4k-2-\epsilon}\big[V_1+2k(\psi+V_0)\big]^2\\
&\qquad-2\tau^{2k-1-\epsilon}\nabla\omega_0\cdot\big[\overline{V}+\tau\nabla\psi+2\tau(\ln\tau)(\psi+V_0)\nabla k\big]\\
&\qquad-\tau^{4k-2-\epsilon}\big[\overline{V}+\tau\nabla\psi+2\tau(\ln\tau)(\psi+V_0)\nabla k\big]^2\Big\}\\
&+\tau^{-\epsilon}\big[\exp(-2\varphi-2\alpha\tau^{2-2k}-2\tau^{2-2k+\epsilon}U_0)(\nabla\omega_0)^2-(2-2k)^2\alpha\big],\\
(1-\tau^2)DV_1 =& -2k(1-\tau^2)V_1+2\tau(\ln\tau)\nabla k\cdot\overline{V}\\
&+2(1-\tau^2)\tau^{2-2k+\epsilon}\big(U_1+(2-2k+\epsilon)U_0\big)\big(V_1+2k(\psi+V_0)\big)\\
&+\tau^2\big[V_1+2k(\psi+V_0)+2(\ln\tau)(\psi+V_0)\Delta k+2(\ln\tau)\nabla k\cdot\nabla\psi+\Delta\psi\big]\\
&+\tau^{2-2k}\big[\Delta\omega_0+2(1-\tau^2)(2-2k)\alpha\big(V_1+2k(\psi+V_0)\big)\big]\\
&+\tau\big[(1-x^2)V_{2x}-xy(V_{2y}+V_{3x})+(1-y^2)V_{3y}-2xV_2-2yV_3\big] \qquad\qquad (47b)\\
&-2\tau^{2-2k}\nabla\omega_0\cdot\big[(\ln\tau)\nabla k+\nabla\varphi\big]-2\tau^{3-4k+\epsilon}\nabla\omega_0\cdot\overline{U}\\
&-2\tau^{4-4k}\nabla\omega_0\cdot\big[\nabla\alpha-2(\ln\tau)(\alpha+\tau^\epsilon U_0)\nabla k\big]\\
&-2\tau^{2-2k}\big[\tau^{2k-1}\nabla\varphi+\tau\nabla\alpha-2\tau(\ln\tau)(\alpha+\tau^\epsilon U_0)\nabla k+\overline{U}\big]\\
&\qquad\cdot\big[2\tau(\ln\tau)(\psi+V_0)\nabla k+\tau\nabla\psi+\overline{V}\big],
\end{aligned}
$$

We choose $\alpha := (2-2k)^{-2}(\nabla\omega_0)^2\exp(-2\varphi)$ to cancel the $\tau^{-\epsilon}$ factor in the last term on the right of (47a). That term will then be of order $\tau^{2-2k-\epsilon}$, so the order of the $(\nabla\omega_0)^2$ term is unchanged from the previous case (45b).

The problematic exponents of τ are now $2-2k-\epsilon$, $3-4k+\epsilon$ and $2k-1-\epsilon$. We have to choose the number $\epsilon>0$ such that $4k-3<\epsilon<2k-1$, but we also need to ensure that $2-2k-\epsilon>0$, which is compatible with $4k-3<\epsilon$ if and only if $1/2<k<5/6$. In that case, we may apply the arguments of section 4.1 and 4.2 to show that a regular solution exists.

Contrary to what was claimed in [38], it is not possible to cover the whole range $1/2<k<1$ since we can only choose α such that the last term in (47a) vanishes to first order in τ. In equation (27) of [38] the corresponding higher order terms have been left out.

We can however cover small intervals of k closer to 1 by repeating the method above. Replacing the expansion (25a) by (46) is equivalent to performing the transformation $u\mapsto\alpha\tau^{2-2k-\epsilon}+\tau^{2-2k}u$. Repeating this transformation i times, where each α is chosen

such that the leading order coefficient of $(\nabla\omega_0)^2$ is cancelled at each stage, gives a system with positive powers of τ if and only if

$$1-\frac{1}{2i}<k<1-\frac{1}{2i+4} \tag{48}$$

and

$$1-2(i+1)(1-k)<\epsilon<\min\{2-2k,1-2i(1-k)\}. \tag{49}$$

The interval $(1/2,1)$ is covered by the infinite sequence of intervals $(1-1/2i,1-1/(2i+4))$. The expansion of Z for a given i is

$$Z(\tau,\theta)=k(\theta)\ln\tau+\varphi(\theta)+\sum_{j=1}^{i}\alpha_j(\theta)\tau^{(2-2k)j}+\tau^{(2-2k)i+\epsilon}u(\tau,\theta). \tag{50}$$

Note that as k tends to 1, we have to include an increasing number of terms in the expansion (50) of Z. This might seem to be contradictory since we keep the original expansion of ω, and the higher order terms of ω should affect Z at some finite order. But for any given k, the number of terms in (50) is finite since i is bounded above by (48). Also, the exponent of τ in the terms containing α_j is always between 0 and 1, so higher order contributions are still encoded in u.

The above discussion motivates the following theorem.

Theorem 4.5. *If k, φ, ω_0 and ψ are smooth functions of θ and k satisfies* (48) *for all θ and some natural number i, there exists a solution of Einstein's equations* (11) *in a neighbourhood $\mathcal{U}$ of $\theta=0$ of the form* (50) *and* (25b) *where ϵ satisfies* (49) *and u and v are regular and tend to* 0 *as $t\to 0$. Given the form of the expansion and a choice of ϵ, the solution is unique.*

5 Discussion

We have constructed families of solutions to the evolution equations of Gowdy spacetimes with $\mathbf{S}^2\times\mathbf{S}^1$ or $\mathbf{S}^3$ spatial topology. When the asymptotic velocity is between 0 and 1 $(0<k<1)$ we obtain solutions depending on four free functions, which is the same number as in the general solution, while outside that range the solutions include only three free functions. The solutions are asymptotically velocity dominated by construction. Unfortunately, for regular asymptotically velocity dominated solutions of the full Einstein equations the asymptotic velocity must be -1 or 3 $(k=\pm 2)$ at the axes. This can also be checked directly by inserting the asymptotic expansions of the solutions into the constraints (17). Since both analytical and numerical arguments indicate that the velocity is between 0 and 1 for $\mathbf{T}^3$ Gowdy, and these results may be applied on sets not containing the axes, it seems that there will be spikes at the axes in general situations.

There are two possible interpretations. Firstly, there is the possibility of false spikes, which result from an inappropriate choice of parametrisation of the metric. This is also what makes our solutions different from the numerical solutions [15]. The parametrisation problem may be traced to different choices of parametrisations of the hyperbolic plane. It might be possible to choose this parametrisation differently such as to avoid coordinate singularities. This is of equal importance in the study of the dynamics of $\mathbf{T}^3$ Gowdy, in particular when trying to verify the numerical observation that the velocity is eventually driven below 1 in a mathematically rigorous way.

Secondly, there is the possibility of true spikes at the axes. In [39], $\mathbf{T}^3$ Gowdy solutions with spikes are constructed by applying suitable transformations to a given solution with velocity between 0 and 1. At first sight it seems possible to do something similar for the other topologies. The transformations used in [39] are inversion, which interchanges the Killing vectors, and a Gowdy-to-Ernst transformation. A combination of these transformations preserve the evolution equations in the $\mathbf{S}^2 \times \mathbf{S}^1$ and $\mathbf{S}^3$ cases as well. In fact, the equations for Z and ω are the same as the equations for $-P$ and Q (in the notation of section 2.1). But P is singular at the axes, so this method of constructing solutions with spikes does not work in our case.

Finally, some remarks on the asymptotic behaviour of the curvature is in order. Using the asymptotic expansions for our solutions, (25) or (31), it is straightforward to calculate that a necessary condition for the Kretschmann scalar $R_{ijkl}R^{ijkl}$ to be bounded is that the asymptotic velocity is ± 1 ($k = 0$ or $k = 2$), which is in agreement with the result for the polarised models described in section 2.4.1. All of the black hole solutions mentioned in section 2.4.2 are of course extendible through the horizon, and the asymptotic velocity is indeed ± 1 there. While such non-generic solutions can be ignored from the cosmological point of view, they might be interesting by analogue to black hole interiors. One interesting question that remains to be answered is under which circumstances general Gowdy models can be extended through a compact Cauchy horizon. In particular, is analyticity necessary as in the polarised case [12]?

To conclude, it seems that for more progress on Gowdy $\mathbf{S}^2 \times \mathbf{S}^1$ and $\mathbf{S}^3$ spacetimes to be made, a better understanding and handling of the spikes are needed. If this can be done for $\mathbf{T}^3$ Gowdy, it should be possible to adapt the techniques to the other topologies using some of the arguments of this paper.

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